Dynamic Thermal Networks. A Methodology to Account for Time-dependent Heat Conduction

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ABSTRACT. Thermal networks provide a handy and instructive way to represent steady-state heat flow processes. Time-dependent thermal flow is much more difficult to handle. The paper outlines a general methodology and theory for time-dependent thermal processes. The method requires that the heat flows through the boundary surfaces are calculated for a unit step change at one surface and zero temperature at the other surfaces. The relations between surface temperatures and heat flows for any time-dependent process are obtained by superposition of the step responses. The theory provides nice insight into the memory effects for time-dependent heat flow. The mathematically exact temperature-flow relations between the surfaces are represented graphically as a dynamic thermal network. The heat flow between two nodes is given by the steady-state conductance times a suitable mean value of the difference of preceding node temperatures. A new conductance has to be added for each surface. This part involves the difference between the actual node temperature and a mean value of preceding temperatures. There are transmittive heat fluxes between all pairs of nodes, and an absorptive flow component at each node.

1 INTRODUCTION

Thermal networks provide a handy and instructive way to represent *steady-state* heat flow processes in cases when the (solid) heat flow region is bounded by a moderate number of surfaces with constant temperatures. Time-dependent thermal flow is much more difficult to handle. An example is the time-variable heat flow out from a building through walls, roof, foundation and surrounding ground. The heat loss depends on actual indoor and outdoor temperatures, and on the preceding sequences of values.

The boundary fluxes depend in general on the *temperature history* of all the boundary surfaces until the considered time. Figure 1 shows a three-surface problem. The indoor, outdoor and attic (air) temperatures are time-dependent. The three sequences of boundary temperatures determine the heat flow process and in particular the three time-dependent boundary heat fluxes. Figure 2 illustrates a general heat conduction problem in solid regions bounded by three surfaces. Our main interest is to determine the three boundary fluxes at any time. The fluxes will depend on the actual boundary temperatures and on the preceding values.

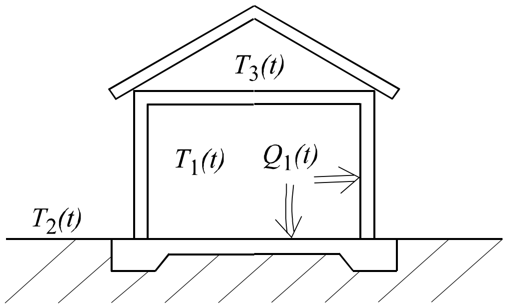


Figure 1.Heat flow problem for a building with varying indoor, outdoor and attic temperatures.

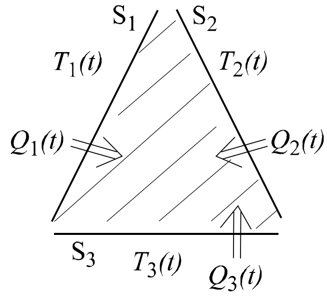


Figure 2. General heat flow problem bounded by three boundary surfaces.

The paper outlines the present state of development of a general theory for time-dependent thermal processes, (Claesson 2002a). The linear heat flow process is prompted by the temperatures at a few boundary surfaces (indoor, outdoor, etc.). It will be shown that it is possible to extend *the concept of thermal networks* to time-dependent cases. A few new concepts are necessary, but the representation is surprisingly straightforward. The starting point is the wellknown technique of superposition of step-response solutions.

An application concerning the heat loss dynamics of walls is presented in a companion paper (Wentzel, Claesson 2003) to this conference. Another application concerning the three-dimensional heat loss dynamics for a whole building with ceiling, walls and foundation is presented in (Wentzel 2002) and (Wentzel 2003). Here, we use the convenient term *dynamic* to signify a *time-dependent* thermal process.

There is a vast literature with many publications in this field of calculating and analyzing boundary heat fluxes. The basic mathematics is found in (Carslaw and Jaeger 1959). The response factor method introduced by Mitalas in the late sixties is summarized together with a large list of references in (ASHREA 1997). Other important papers are (Davies 1997), where the analysis is based on ramp responses, and (Kossecka and Kosny 1998), where so-called thermal structure factors for composite walls are used.

2 HEAT CONDUCTION PROBLEM

The region of heat conduction is bounded by a few boundary surfaces with prescribed temperatures. Each temperature is constant over its surface, and it may be any function of time. The thermal properties may vary in any way throughout the region. A basic limitation is that the heat flow problem is linear, so that the principle of superposition is applicable.

$$\nabla \cdot (\lambda \, \nabla T) = \rho c \cdot \frac{\partial T}{\partial t}$$

$$T_1(t) - T|_{\mathrm{S}_1} = \frac{\lambda}{\alpha_1} \frac{\partial T}{\partial n}\bigg|_{\mathrm{S}_1}$$

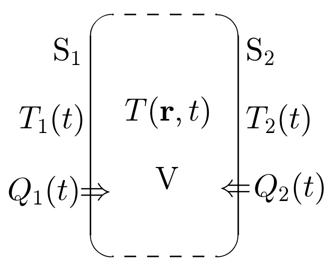


Figure 3. Linear heat flow problem (for the case of two boundary surfaces).

The normal heat flux at surface S_1 equals the temperature difference over the surface layer times the surface heat transfer coefficient α_1.

3 DYNAMIC THERMAL NETWORK; TWO SURFACES

We consider first a heat flow problem with two boundary surfaces. Our main interest is the dynamic relations between the boundary heat fluxes and temperatures. In Section 4 we will consider certain step-response solutions. By suitable superposition, where the boundary temperatures are considered as integrals of infinitesimal step, we get integral formulas for the boundary fluxes, (9). These relations may be rewritten in the following general way:

$$Q_1(t) = K_1 \left[T_1(t) - \overline{T}_{1\mathrm{a}}(t) \right] + K_{12} \left[\overline{T}_{1\mathrm{t}}(t) - \overline{T}_{2\mathrm{t}}(t) \right],$$

$$Q_2(t) = K_2 \left[T_2(t) - \overline{T}_{2\mathrm{a}}(t) \right] + K_{12} \left[\overline{T}_{2\mathrm{t}}(t) - \overline{T}_{1\mathrm{t}}(t) \right]. \tag{1}$$

The (steady-state) *thermal conductance* between the two surfaces is K_{12} (W/K). The inverse $R_{12} = 1/K_{12}$ is the thermal resistance between the boundary surfaces. The factor K_1 (W/K) is the *surface thermal conductance* for surface S_1. It is equal to the surface area A_1 times the surface heat transfer coefficient: $K_1 = A_1 \cdot \alpha_1$.

The right-hand terms of the dynamic relations involve thermal conductances multiplied by temperature differences. The following notations are used:

$$\overline{T}_{1\mathrm{a}}(t) = \int_0^\infty \kappa_{1\mathrm{a}}(\tau) \cdot T_1(t-\tau)\, d\tau,$$

$$\overline{T}_{1\mathrm{t}}(t) = \int_0^\infty \kappa_{12}(\tau) \cdot T_1(t-\tau)\, d\tau,$$

$$\overline{T}_{2\mathrm{a}}(t) = \int_0^\infty \kappa_{2\mathrm{a}}(\tau) \cdot T_2(t-\tau)\, d\tau,$$

$$\overline{T}_{2\mathrm{t}}(t) = \int_0^\infty \kappa_{12}(\tau) \cdot T_2(t-\tau)\, d\tau. \tag{2}$$

The integrals are average values of the boundary temperatures backward in time. The *weight functions* $\kappa_{1\mathrm{a}}(\tau)$, $\kappa_{12}(\tau)$ and $\kappa_{2\mathrm{a}}(\tau)$ are defined below.

Let us first discuss the structure of these relations. The boundary fluxes (1) may be divided into two parts:

$$Q_1(t) = Q_1^{\mathrm{ab}}(t) + Q_{1\to 2}^{\mathrm{tr}}(t),$$

$$Q_2(t) = Q_2^{\mathrm{ab}}(t) - Q_{1\to 2}^{\mathrm{tr}}(t). \tag{3}$$

Here, we define

$$Q_1^{\mathrm{ab}}(t) = K_1 \cdot \left[T_1(t) - \overline{T}_{1\mathrm{a}}(t) \right],$$

$$Q_2^{\mathrm{ab}}(t) = K_2 \cdot \left[T_2(t) - \overline{T}_{2\mathrm{a}}(t)\right],$$

$$Q_{1\to 2}^{\mathrm{tr}}(t) = K_{12} \cdot \left[\overline{T}_{1\mathrm{t}}(t) - \overline{T}_{2\mathrm{t}}(t)\right],$$

$$Q_{2\to 1}^{\mathrm{tr}}(t) = -Q_{1\to 2}^{\mathrm{tr}}(t). \qquad (4)$$

We will call them *absorptive* and *transmittive* heat fluxes.

Figure 4 illustrates this division of the boundary fluxes. The absorptive component of the heat flux through S_1 is equal to the surface conductance K_1 multiplied by the difference between the *present* temperature $T_1(t)$ and an average $\overline{T}_{1\mathrm{a}}(t)$ backward in time of the *same* surface temperature. There is a corresponding absorptive component for surface S_2. The transmittive flux is the same for both boundary fluxes with opposite signs. It is equal to the *steady-state conductance* K_{12} between S_1 and S_2 multiplied by an *average* temperature difference between the surfaces, (4).

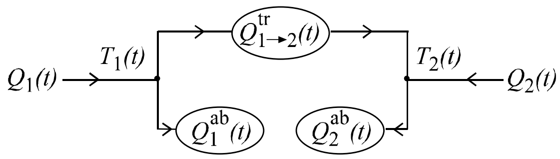


Figure 4. Division of the boundary fluxes into absorptive and transmittive parts in the two-surface case.

The absorptive flux $Q_1^{\mathrm{ab}}(t)$ depends on $T_1(t)$ but not on the other surface temperature $T_2(t)$. It is equal to the total heat influx (at both sides) when $T_1(t)$ acts at S_1, while the temperature is *zero* at the other surface. See Figure 5, left. This follows from (1) with $T_2 = 0$ by adding the two equations. There is a corresponding interpretation for $Q_2^{\mathrm{ab}}(t)$.

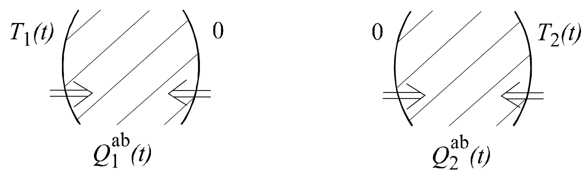


Figure 5. Interpretation of the two absorptive fluxes.

The transmittive flux depends only on the difference $\overline{T}_{1\mathrm{t}}(t) - \overline{T}_{2\mathrm{t}}(t)$ between the boundary temperatures. It is obtained as the flux out from S_2, when the temperature difference $T_1(t) - T_2(t)$ acts at S_1 and $T = 0$ at S_2. See Figure 6. This follows from the equation for $Q_2(t)$ in (1) by putting $T_2 = 0$ and replacing T_1 by $T_2 - T_1$.

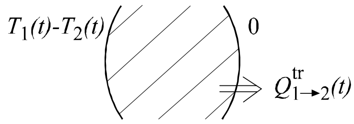


Figure 6. Interpretation of the the transmittive flux.

The basic relations (1) with the temperature averages (2) may be interpreted as a *dynamic thermal network*. Figure 7 shows a suggested graphical representation of these dynamic (or time-dependent) relations.

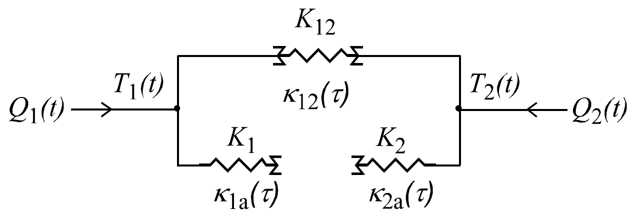
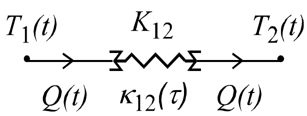


Figure 7. Graphical representation of the relations (1)-(2) as a dynamic thermal network for a two-surface problem.

The transmittive component is represented by the conventional resistance symbol with the thermal conductance $K_{12} = 1/R_{12}$ written above it. To the resistance or conductance symbol, we add *summation signs* on both sides. The signs signify that we take an average of the node temperatures according to (2). The left-hand summation sign is reversed to indicate the symmetric character of the flux and that summation concerns the values at the left-hand node. The weight function $\kappa_{12}(\tau)$ may be written below the resistance symbol. The symbol represents graphically the following equation for the heat flux:

$$Q(t) = K_{12} \cdot \int_0^\infty \kappa_{12}(\tau)\left[T_1(t-\tau) - T_2(t-\tau)\right]\, d\tau$$

$\Leftrightarrow$

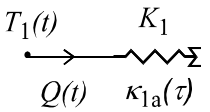


(5)

The two absorptive components are represented by the resistance symbol with the surface thermal conductance (K_1 and K_2) written above. The weight functions $\kappa_{1\mathrm{a}}(\tau)$ and $\kappa_{2\mathrm{a}}(\tau)$ may be written below the resistance symbols. A summation sign is added at the *free end* after the surface conductance. There is not any summation sign on the node side, since the *present* node temperature is to be used at this side. This new symbol represents graphically the following equation for the heat flux:

$$Q(t) = K_1 \cdot \left[T_1(t) - \int_0^\infty \kappa_{1\mathrm{a}}(\tau) T_1(t-\tau)\, d\tau \right]$$

$$\Leftrightarrow \qquad (6)$$

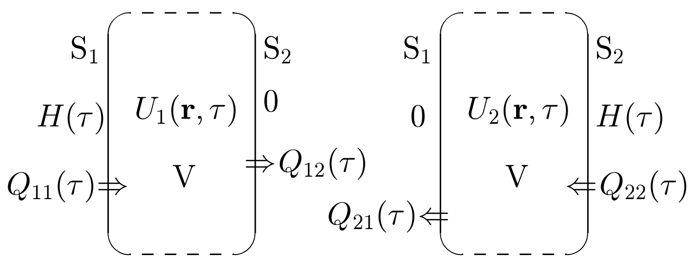


4 BASIC STEP-RESPONSE SOLUTIONS

We will use certain basic dynamic solutions from which general solutions are obtained by superposition. The boundary temperatures $T_1(t)$, $T_2(t)$, etc., are the input in our thermal processes. The basic output is the boundary heat fluxes $Q_m(t)$.

The *simplest possible* time-dependent input is a unit temperature step at a certain time. We will here consider the very basic solutions for *a unit step* in the air immediately outside one boundary surface. All temperatures are zero for $\tau < 0$. At this time one boundary temperature is changed to 1 and kept at this value for all times $\tau > 0$. All other boundary temperatures are zero (for all times). The initial temperature in V (at $\tau = 0$) is zero. The solutions for these boundary steps will be called the *basic step-response solutions*. We use τ and not t as time variable in the step-response solutions. Then the weight functions have τ as independent variable, while the backward boundary temperatures are taken for $t - \tau$.

Figure 8 illustrates the basic step-response problems with the solutions $U_1(\mathbf{r},\tau)$ and $U_2(\mathbf{r},\tau)$ for a two-surface problem. For each problem there are two boundary fluxes as shown in the figure.



$$H(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

Figure 8 The two basic step-response problems and the definition of boundary fluxes in the two-surface case.

4.1 *Basic step-response boundary fluxes*

The two cross-fluxes $Q_{12}(\tau)$ and $Q_{21}(\tau)$ are equal (for all times) due to a wellknown general symmetry principle (see Claesson 2002a). In a case with two boundary surfaces, there are three response functions:

$$Q_{11}(\tau), \quad Q_{22}(\tau), \quad Q_{12}(\tau) = Q_{21}(\tau). \qquad (7)$$

At the time $\tau = 0$, we impose the temperature 1 outside the surface layer. The temperature difference over the surface heat transfer coefficient is $1 - 0$. This means that the first two fluxes, the *admittive* step-response fluxes, start from their respective surface thermal conductivity (K_1 and K_2). These fluxes decrease monotonously to the steady-state flux K_{12} between the surfaces after long time. The cross-flux, or *transmittive* flux, $Q_{12}(\tau) = Q_{21}(\tau)$ starts from zero and increases to K_{12}. See figure 9. We will also, in accordance with figures 5 and 8, use the following *absorptive* step-response fluxes

$$Q_{1\mathrm{a}}(\tau) = Q_{11}(\tau) - Q_{12}(\tau),$$
$$Q_{2\mathrm{a}}(\tau) = Q_{22}(\tau) - Q_{12}(\tau). \qquad (8)$$

The absorptive fluxes start from K_1 and K_2 for $\tau = 0$ and decrease monotonously to zero for large τ.

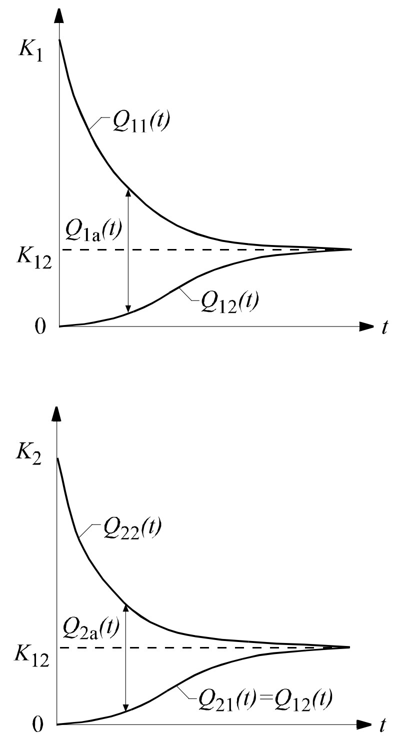


Figure 9. Character of the basic step-response boundary fluxes for a step at S_1, top, and at S_2, bottom.

4.2 *Superposition formula. Weight functions*

We may obtain the solution for any boundary temperatures $T_1(t)$ and $T_2(t)$ by a suitable superposition of the basic step-response solutions, (Carslaw and Jaeger 1959). The Duhamel superposition formulas may be written in different ways. We have the following general, *mathematically exact* equations, (Claesson 2002a), which are particularly suited for our purposes:

$$Q_1(t) = K_1 \cdot T_1(t) + \int_0^\infty \frac{dQ_{1a}}{d\tau} \cdot T_1(t-\tau)d\tau +$$

$$\int_0^\infty \frac{dQ_{12}}{d\tau} \cdot [T_1(t-\tau) - T_2(t-\tau)]\, d\tau,$$

$$Q_2(t) = K_2 \cdot T_2(t) + \int_0^\infty \frac{dQ_{2a}}{d\tau} \cdot T_2(t-\tau)d\tau +$$

$$\int_0^\infty \frac{dQ_{12}}{d\tau} \cdot [T_2(t-\tau) - T_1(t-\tau)]\, d\tau. \quad (9)$$

We now define *weight functions* in the following way:

$$\frac{dQ_{1a}(\tau)}{d\tau} = -K_1 \cdot \kappa_{1a}(\tau),$$

$$\frac{dQ_{12}(\tau)}{d\tau} = K_{12} \cdot \kappa_{12}(\tau),$$

$$\frac{dQ_{2a}(\tau)}{d\tau} = -K_2 \cdot \kappa_{2a}(\tau). \quad (10)$$

Insertion of these expressions in (9) gives the basic equations (1).

The weight functions are positive (or zero). The integrals of the weight functions are equal to 1, since $Q_{1a}(\tau)$ varies from K_1 to zero and $Q_{12}(\tau)$ from zero to K_{12}. We have

$$\int_0^\infty \kappa_I(\tau)d\tau = 1 \quad \kappa_I(\tau) \geq 0 \qquad I = 1\text{a},\ 12,\ 2\text{a}. \quad (11)$$

The absorptive weight functions $\kappa_{1a}(\tau)$ and $\kappa_{2a}(\tau)$ decrease steadily from large values at small τ to zero at infinity. The transmittive weight function $\kappa_{12}(\tau)$ increases from zero to a maximum and then decreases to zero at infinity. It has a bell-shaped form. See Figure 10.

An important application of the response theory is composite walls with time-dependent one-dimensional heat flow. A new method for rapid and very exact calculation of the step-response solutions is presented in Claesson (2002b). Laplace and generalized Fourier techniques are used in combination. The response and weight functions are quite readily obtained for any composite wall with very high accuracy.

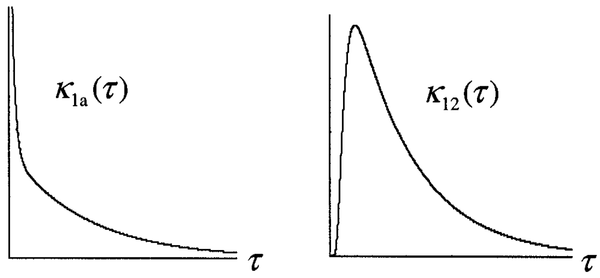


Figure 10. Character of absorptive and transmittive weight functions.

5 DYNAMIC THERMAL NETWORK; THREE SURFACES

It is straight-forward to generalize the above relations to cases with any number of surfaces. However, the theory is most useful for problems with a moderate number of boundary surfaces, each having a prescribed time-dependent temperature.

Figure 2 illustrates the general three-surface problem. The corresponding dynamic thermal network for a three-surface problem is shown in Figure 11. There are three surface conductances K_1, K_2 and K_3, and three (steady-state) thermal conductances K_{12}, K_{13} and K_{23} between the boundary surfaces. For constant boundary temperatures, the network becomes identical with the ordinary steady-state network, since the absorptive parts and the summations backward in time vanish.

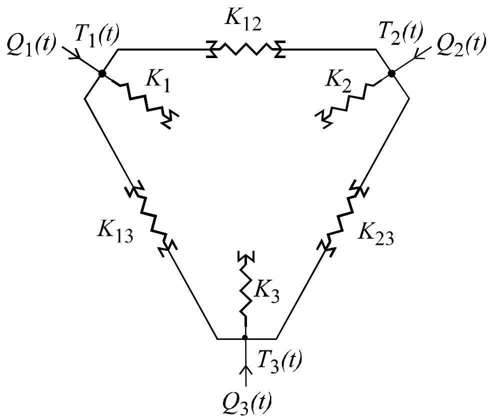


Figure 11. Dynamic thermal network for a three-surface problem.

The two basic step-response problems for the two-surface case are illustrated in Figure 8. We have now three step-response problems with the solutions $U_1(\mathbf{r},\tau)$, $U_2(\mathbf{r},\tau)$ and $U_3(\mathbf{r},\tau)$. The boundary temperature for $U_1(\mathbf{r},\tau)$ is zero outside S_2 and S_3, while there is a unit step $H(\tau)$ outside S_1. The admittive response influx at S_1 is $Q_{11}(\tau)$, and the response cross-fluxes out through S_2 and S_3 are $Q_{12}(\tau)$ and $Q_{13}(\tau)$, respectively. There are corresponding response fluxes $Q_{22}(\tau)$, $Q_{21}(\tau)$ and $Q_{23}(\tau)$ for U_2, and $Q_{33}(\tau)$, $Q_{31}(\tau)$ and $Q_{32}(\tau)$ for U_3. There are in this case 3+3 basic response fluxes due to the symmetry of the cross-fluxes ($Q_{12}(\tau) = Q_{21}(\tau)$, $Q_{13}(\tau) = Q_{31}(\tau)$ and $Q_{23}(\tau) = Q_{23}(\tau)$). The absorptive fluxes are defined as

$$Q_{1a}(\tau) = Q_{11}(\tau) - Q_{12}(\tau) - Q_{13}(\tau). \tag{12}$$

We have six weight functions defined as in (10). We have for example:

$$\frac{dQ_{3a}(\tau)}{d\tau} = -K_3 \cdot \kappa_{3a}(\tau). \tag{13}$$

$$\frac{dQ_{13}(\tau)}{d\tau} = K_{13} \cdot \kappa_{13}(\tau). \tag{14}$$

There are three absorptive weight functions $\kappa_{1a}(\tau)$, $\kappa_{2a}(\tau)$ and $\kappa_{3a}(\tau)$, and three transmittive weight functions $\kappa_{12}(\tau)$, $\kappa_{13}(\tau)$ and $\kappa_{23}(\tau)$. They are all positive (or zero), and their integrals are all equal to 1 in accordance with (11).

The general formulas (9), which relate the boundary fluxes to the boundary temperatures are readily extended to the three-surface case. For example, we have for $Q_1(t)$

$$Q_1(t) = K_1 \cdot T_1(t) + \int_0^\infty \frac{dQ_{1a}}{d\tau} \cdot T_1(t-\tau)d\tau +$$
$$\int_0^\infty \frac{dQ_{12}}{d\tau} \cdot [T_1(t-\tau) - T_2(t-\tau)]\, d\tau +$$
$$\int_0^\infty \frac{dQ_{13}}{d\tau} \cdot [T_1(t-\tau) - T_3(t-\tau)]\, d\tau. \tag{15}$$

The heat flux through surface S_1 may now be written

$$Q_1(t) = K_1 \left[T_1(t) - \overline{T}_{1a}(t)\right] +$$
$$K_{12}\left[\overline{T}_{1t2}(t) - \overline{T}_{2t1}(t)\right] +$$
$$K_{13}\left[\overline{T}_{1t3}(t) - \overline{T}_{3t1}(t)\right]. \tag{16}$$

There are corresponding equations for $Q_2(t)$ and $Q_3(t)$. The corresponding thermal network is shown in Figure 11. The six average temperatures $\overline{T}_{mtm'}$ ($m' \neq m$) are defined in the same way as in (2). We have for example

$$\overline{T}_{1t2}(t) = \int_0^\infty \kappa_{12}(\tau) \cdot T_1(t-\tau)\, d\tau,$$

$$\overline{T}_{2t1}(t) = \int_0^\infty \kappa_{12}(\tau) \cdot T_2(t-\tau)\, d\tau. \tag{17}$$

Here, we use the index mtm' to denote the mean of temperature $T_m(t-\tau)$ with the transmittive weight function $\kappa_{mm'}(\tau)$ ($m \neq m'$).

The above theory is applicable for any number of boundary surfaces M. For $M = 1$ we have a single boundary surface. There is a single step-response function $Q_{11}(\tau) = Q_{1a}(\tau)$, and a corresponding absorptive weight factor $\kappa_{1a}(\tau)$. There is not any transmittive part. The thermal network consists of an absorptive component only as shown in (6).

In the general case with M boundary surfaces, there are M step-response problems and $M \cdot M$ step-response fluxes $Q_{mm'}(\tau)$. The cross-fluxes $Q_{mm'}(\tau)$ and $Q_{m'm}(\tau)$ are equal ($m \neq m'$). We get M absorptive and $M\cdot(M-1)/2$ transmittive step-response fluxes. These fluxes give M absorptive weight functions $\kappa_{ma}(\tau)$ and $M\cdot(M-1)/2$ transmittive weight functions $\kappa_{mm'}(\tau)$ as in (10) and (13)-(14).

6 DISCRETE APPROXIMATION

In a numerical solution, we must use a discrete approximation. Let $h > 0$ be the time step. The time interval under consideration, $nh - h \leq t \leq nh$, has index n. The preceding intervals, $nh - \nu h - h \leq t \leq nh - \nu h$, are enumerated backwards in time ($\nu = 1,\ 2,\ ...$). We may consider a *linear* temperature variation during each time step with the temperature $T_{1,n-\nu} = T_1(nh - \nu h)$ at the righthand end point of interval $n - \nu$. The heat flux at the considered time is $Q_{1,n} = Q_1(nh)$.

The relations between boundary temperatures and boundary heat fluxes become in the considered discrete approximation, (Claesson 2002a):

$$Q_{1,n} = \bar{K}_1\left[T_{1,n} - \overline{T}_{1a,n}\right] + K_{12}[\overline{T}_{1t,n} - \overline{T}_{2t,n}],$$

$$Q_{2,n} = \bar{K}_2\left[T_{2,n} - \overline{T}_{2a,n}\right] + K_{12}[\overline{T}_{2t,n} - \overline{T}_{1t,n}]. \tag{18}$$

In the discrete form of (2), we get the following average values of the boundary temperatures:

$$\overline{T}_{1a,n} = \sum_{\nu=1}^{\nu_s} \kappa_{1a,\nu} T_{1,n-\nu}, \quad \overline{T}_{2a,n} = \sum_{\nu=1}^{\nu_s} \kappa_{2a,\nu} T_{2,n-\nu}\ ,$$

$$\overline{T}_{1t,n} - \overline{T}_{2t,n} = \sum_{\nu=0}^{\nu_s} \kappa_{12,\nu}\left[T_{1,n-\nu} - T_{2,n-\nu}\right]. \tag{19}$$

The relations (18) and (19) are the discrete form of (1) for the dynamic thermal network of Figure 2. The surface conductances are replaced by modified surface conductances and the weight functions by weight factors for each time step. The integration to infinity in the temperature averages (2) must be limited to a finite value τ_s at which time the weight functions are zero with a sufficient accuracy. Then steady-state conditions are attained within the considered accuracy. The summations are performed up to a large $\nu = \nu_s$ with $\nu_s \simeq \tau_s/h$.

We use a time-step *average* $\bar{Q}_I(\tau)$ of the response functions $Q_I(\tau)$:

$$\bar{Q}_I(\tau) = \frac{1}{h}\int_{\tau}^{\tau+h} Q_I(\tau')\, d\tau', \quad I = 1\text{a},\ 12,\ 2\text{a}. \tag{20}$$

The *modified* surface conductances $\bar{K}_1$ and $\bar{K}_2$ are:

$$\bar{K}_1 = \bar{Q}_{1\text{a}}(0), \qquad \bar{K}_2 = \bar{Q}_{2\text{a}}(0). \tag{21}$$

From (2) with piece-wise linear boundary temperatures, we get the weight factors (Claesson 2002a):

$$\kappa_{1\text{a},\nu} = \frac{\bar{Q}_{1\text{a}}(\nu h - h) - \bar{Q}_{1\text{a}}(\nu h)}{\bar{K}_1}.$$

$$\kappa_{12,\nu} = \frac{\bar{Q}_{12}(\nu h) - \bar{Q}_{12}(\nu h - h)}{K_{12}},$$

$$\kappa_{2\text{a},\nu} = \frac{\bar{Q}_{2\text{a}}(\nu h - h) - \bar{Q}_{2\text{a}}(\nu h)}{\bar{K}_2}. \tag{22}$$

The equations (18)-(22) are *exact* for the considered piece-wise linear boundary temperatures.

These equations are the same as the ones obtained with the response factor method (Mitalas 1978 and others, ASHREA Handbook of Fundamentals, 1985). The difference is that we start from the response for a unit step, while the other method considers the response for a triangular pulse with a fixed width $(2 \cdot h)$. An advantage with the presented approach is that the basic response is independent of any time step. We use and store the basic response functions for the particular heat flow problem. The response factors for any discrete approximation are obtained from formulas of the above type (Claesson 2002a). The integral of the step response gives the response for a ramp temperature ($T_1(\tau) = \tau$). The triangular pulse may be obtained from three superimposed ramps (Davies 1997, Claesson 2002a). This means that the response for a triangular pulse is obtained from integrals of three step responses. It is easier to represent the step-response functions than the more complex triangular-response functions with a given accuracy.

7 THERMAL NETWORKS. EXAMPLES

The use of dynamic thermal networks are illustrated by two examples. The first examples concerns the heat balance for the indoor temperature. The second example is a heat balance for a crawl space.

7.1 *Indoor temperature*

We consider a building with an indoor temperature $T_1(t)$ and a given outdoor temperature $T_2(t)$. There is a given heat input $Q_\text{h}(t)$ (W) and a given heat gain from solar radiation $Q_\text{r}(t)$. The heat input from ventilation is given by a prescribed time-dependent ventilation conductance $K_\text{v}(t)$ (W/K). The sum of these heat inputs is equal to the heat flux $Q_1(t)$ at the inside boundary of the building. Using (1), we have the heat balance

$$Q_1(t) = K_1\left[T_1(t) - \overline{T}_{1\text{a}}(t)\right] + K_{12}\left[\overline{T}_{1\text{t}}(t) - \overline{T}_{2\text{t}}(t)\right]$$

$$= Q_\text{h}(t) + Q_\text{r}(t) + K_\text{v}(t)\left[T_2(t) - T_1(t)\right]. \tag{23}$$

The corresponding dynamic thermal network is shown in Figure 12. The absorptive part at the outdoor side S_2 in the general Figure 7 is omitted. We do not need this part for the balance on the inside.

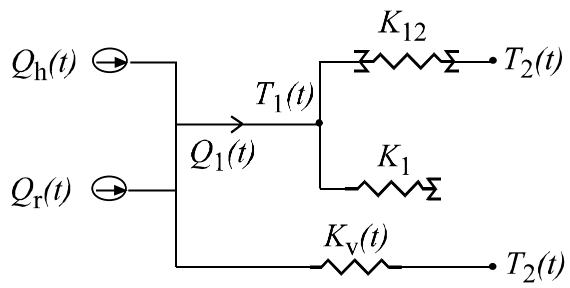


Figure 12 Dynamic thermal network for the indoor temperature with heating, solar radiation and ventilation.

The indoor temperature $T_1(t)$ is readily solved from the heat balance (23):

$$T_1(t) = \frac{Q_\text{h}(t) + Q_\text{r}(t) + K_\text{v}(t)\, T_2(t) + Q_A}{K_1 + K_\text{v}(t)},$$

$$Q_A = K_1\, \overline{T}_{1\text{a}}(t) + K_{12}\left[\overline{T}_{2\text{t}}(t) - \overline{T}_{1\text{t}}(t)\right]. \tag{24}$$

The indoor temperature is equal to the heat influx to the indoor node from heating, solar radiation, ventilation with the outdoor temperature, absorptive influx with absorptive temperature $\overline{T}_{1\text{a}}(t)$ and a transmittive influx with the temperature difference $\overline{T}_{2\text{t}}(t) - \overline{T}_{1\text{t}}(t)$, all divided by the sum of conductances $K_1 + K_\text{v}(t)$.

The absorptive and transmittive average temperatures are discretized in accordance with (18)-(22). The temperature $T_{1,n}$ at time step n is obtained from (24). In the sums, we use preceeding values $T_{1,n-\nu}$ and $T_{2,n-\nu}$ for $\nu = 1, \ldots, \nu_s$. These previous values have to be stored.

The above very simple case with prescribed heating may readily be extended to more complex regimes for $Q_h(t)$, and to more complex networks. The dynamic thermal networks for heat flow in solid regions may in the above way be added and incorporated within the *conceptual framework* of steady-state networks.

7.2 *Crawl-space temperature*

We consider the heat balance in a ventilated crawl space under a building with constant indoor temperature T_1. The outdoor temperature $T_2(t)$ is given. The heat input from ventilation is given by a prescribed ventilation conductance $K_v(t)$.

The heat input from the ventilation is equal to the flux $Q_3(t)$ from the crawl space. Here, we neglect the small heat capacity of the air in the crawl space. There is an absorptive flux and transmittive fluxes to the indoor surface with the temperature T_1 and to the outdoor surface with the temperature $T_2(t)$. We have in accordance with (16)

$$Q_3(t) = K_3\left[T_3(t) - \overline{T}_{3a}(t)\right] + K_{23}\left[\overline{T}_{3t2}(t) - \overline{T}_{2t3}(t)\right] + K_{13}\left[\overline{T}_{3t1}(t) - T_1\right] = = K_v(t)\left[T_2(t) - T_3(t)\right]. \quad (25)$$

The average temperature $\overline{T}_{1t3}(t)$ is equal to the constant value T_1.

The corresponding dynamic thermal network is shown in Figure 13. The absorptive components for the indoor and outdoor nodes, and the transmittive component between them, are omitted from the general Figure 11. We do not need these parts for the crawl-space balance. We also omit the sigma sign towards T_1 in K_{13} since no summation is needed here.

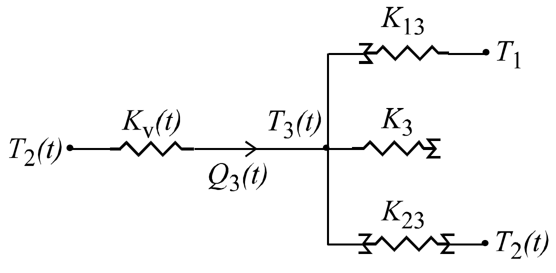


Figure 13 Dynamic thermal network for a ventilated crawl-space temperature. The indoor temperature T_1 is constant.

The indoor temperature $T_1(t)$ is readily solved from the heat balance (25)

$$T_3(t) = \frac{K_{13}\left[T_1 - \overline{T}_{3t1}(t)\right] + K_3\,\overline{T}_{3a}(t) + Q_B}{K_3 + K_v(t)},$$

$$Q_B = K_{23}\left[\overline{T}_{2t3}(t) - \overline{T}_{3t2}(t)\right] + K_v(t)\,T_2(t). \quad (26)$$

It is straightforward to implement the dynamic thermal networks is a computer code or in any mathematical program. The level of complexity is somewhat higher than that of the corresponding steady-state networks. We have to add an absorptive component at each node, and store and use preceding values in the sums for the absorptive and transmittive components.

8 CONCLUDING REMARKS

The paper outlines the a general theory for time-dependent thermal processes where the relations between boundary temperatures and heat fluxes may be represented as a dynamic thermal network. The linear heat flow process is prompted by the temperatures at a few boundary surfaces.

The method requires that the heat flows through these surfaces are calculated for a unit step change at one surface while keeping zero temperature at the other surfaces. The relations between surface temperatures and heat flows for any time-dependent process are obtained by superposition of the basic step responses. This means that step-response flows contain all information required for any particular case.

The mathematically exact temperature-flow relations between the surfaces are represented graphically as a dynamic thermal network. The number of nodes is equal to the number of surfaces. It turns out to be possible to retain the steady-state thermal conductances (or resistances) between the nodes. The heat flow between two nodes is given by the conductance times a suitable mean value of the difference of preceding node temperatures. This mean value over preceding time is determined by a weight function that is proportional to the time-derivative of the corresponding step-response flux.

The other main difference compared to a steady-state network is that a new conductance has to be added at each surface in order to account for the total flux to the node. This part involves the difference between the actual node temperature and a mean value of preceding temperatures at the same node. The weight function for the mean value is determined from a corresponding response function.

We get transmittive heat fluxes between the nodes just as in the steady-state case. At each surface we get a new absorptive heat flux involving the surface conductance. The absorptive flux may be regarded as the heat uptake (from all surfaces) for the given boundary temperature when the other boundary temperatures are zero.

The dynamic networks have the same accuracy as the full (three-dimensional) models used to calculate the step responses, but they will typically be 100 times faster or more on a computer. An example is presented in Svensson and Claesson (1999). The theory provides nice insight into the memory effects for time-dependent heat flow.

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